Triple Integrals

Tuesday, March 16, 2021 12:22 PM

2. We talked about how to calculate double integrals. Now: theory

This time...
- use Riemann sum to explain change of vars
- introduce triple-integrals

Usual Riemann sum:
Idea 0

Problem: $f(x)$ varies as x goes from a to b
Solution: Break up $[a,b]$ into little pieces on which f is $\approx$ a const
(bc f can't change too much on a small enough interval)
- precisely: continuity

- usual way to break up $[a,b]$, is into N intervals, each of the same length
- $I_i = \left[a + \frac{(i-1)(b-a)}{N},\ a + \frac{i(b-a)}{N} \right]$

I is partitioned into the I_i
↳ Def: A partition of I is a way of breaking I into smaller intervals
$I = I_1 \cup I_2 \cup I_3 \cup \dots \cup I_n$
so that the interiors of smaller intervals don't overlap

Note: If $I_i = [a_i, b_i]$, its interval is (a_i, b_i)
- eg $[0,1] = [0, \frac{1}{3}] \cup [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{5}{8}] \cup [\frac{5}{8}, 1]$
- mesh of a partition is max length (I_i)

eg mesh $\frac{3}{8}$

For a region R in $\mathbb{R}$, a partition of R is a decomposition $R = R_1 \cup R_2 \cup \dots \cup R_N$
mesh(partition) = max area (R_i)

eg $R = [a,b] \times [c,d]$
choose M, and have $N = M^2$ little rectangles indexed by $i,j = 1, \dots, M$

$$\left[a + \frac{(i-1)(b-a)}{N},\ a + \frac{i(b-a)}{N} \right] \times \left[c + \frac{(j-1)(d-c)}{M},\ c + \frac{j(d-c)}{M} \right]$$

area of $R_{ij} = \frac{(b-a)(d-c)}{M} =$ mesh

Back to 1-D:
Given a partition $I = I_1 \cup I_2 \cup \dots \cup I_N$

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Given a partition $I = I_1 \cup I_2 \cup \dots \cup I_N$

Choose $x_i \in I_i \ \forall i$

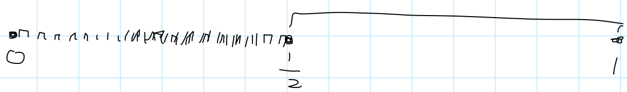
$$\text{Riemann sum} = \sum_{i=1}^{N} f(x_i) \cdot \text{length}(I_i)$$

$$\int_a^b f(x)\,dx = \int_I f(x)\,dx = \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i)$$

- mesh being small $\Rightarrow$ every subinterval is "little enough"

Q/ why can't just take $N \to \infty$?

What if $I = [0,1]$



So divide $[0, \frac{1}{2}]$ into $N-1$ pieces and take $I_N = [\frac{1}{2}, 1]$

then as $N \to \infty$, the # of subintervals $\to \infty$

but the mesh stays $\frac{1}{2}$

$\Rightarrow$ need mesh to approach 0

Precisely $\lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i) \cdot \text{length}(I_i) = L$

means $\forall \varepsilon > 0, \ \exists \delta > 0$ s.t. for any partition $I = I_1 \cup \dots \cup I_N$ of mesh $< \delta$ and any choice of $x_i \in I_i \forall i$

$$\left| L - \sum_{i=1}^{N} f(x_i) \,\text{len}(I_i) \right| < \varepsilon$$

Thm

$\lim_{\text{mesh}\to 0} \sum_{i=1}^{N} \dots$ exists for f continuous from a to b and is the integral as we know it

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2 Dimensions:

Recall a partition of R is a decomp $R = R_1 \cup R_2 \cup \dots \cup R_N$ whose interiors don't overlap.

$$\iint_R f(x,y)\,dx\,dy = \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i, y_i)\, \text{area}(R_i)$$

of the partition

$(x_i, y_i) \in R_i$

eg Divide $[a,b] \times [c,d]$ into rectangles as above.

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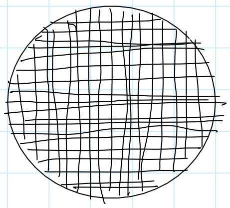
eg Sierpinski's triangle

eg a circle



→ technically need diameters to approach 0

Note: theory of Riemann sums and mesh is theory — use it to prove general facts about integration but don't compute w/ it directly

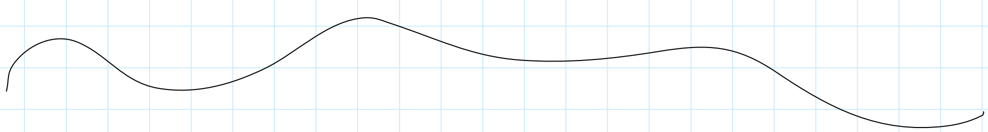
e.g.

$$\left.\iint_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dx\,dy \right] \text{ FUBINI'S THEOREM}$$

$$\iint_{R_1 \cup R_2} f(x,y)\,dx\,dy = \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

if R_1 & R_2 have disjoint interiors

Proof let $L = \iint_{R_1 \cup R_2}$ $\quad L_1 = \iint_{R_1}$ $\quad L_2 = \iint_{R_2}$



Now the Riemann sum over $R_1 \cup R_2$ is the sum of the Riemann sums over each individual region R_1 and R_2

$$\Rightarrow \left| \iint_{R_1\cup R_2} f\,dx\,dy - \iint_{R_1} f\,dx\,dy - \iint_{R_2} f\,dx\,dy \right| < \varepsilon$$

$$\iint_{R_1\cup R_2} f\,dx\,dy - \iint_{R_1} f\,dx\,dy - \iint_{R_2} f\,dx\,dy = 0 \quad \blacksquare$$

Triple Integration

suppose $f: D \to \mathbb{R}$ for $\underset{\text{open}}{D \subseteq} \mathbb{R}^3$ and $R \subseteq D$

Rough idea

$$\iiint_R f\,dx\,dy\,dz = f(x,y,z)\cdot \text{volume}(R)$$

A partition of $R = R_1 \cup \cdots \cup R_N$ has a mesh

$$\iiint_R f\,dx\,dy\,dz = \lim_{\text{mesh}\to 0} \sum_{i=1}^{N} f(x_i, y_i, z_i)\,.\text{volume}(R_i)$$

$(x_i, y_i, z_i) \in R$

↳ calculate in a similar way as in 2-D

eg

$R = [a,b] \times [c,d] \times [e,f]$

$$\iiint_R g(x,y,z)\,dx\,dy\,dz$$

$$= \int_e^f \int_c^d \int_a^b g(x,y,z)\,dx\,dy\,dz$$

$R = \text{unit ball} = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 \le 1 \right\}$

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} f\,dz\,dy$$

↳ easier in spherical coords

Btwn cylindrical & cartesian

Recall

$dx\,dy = r\,dr\,d\theta$

$\Rightarrow dx\,dy\,dz = (dx\,dy)\,dz = (r\,dr\,d\theta)\,dz$

$= r\,dr\,d\theta\,dz$

Center of mass

Suppose we have n objects indexed by $i = 1, \ldots, n$

where the ith object is at location

$\vec{r}_i = (x_i, y_i, z_i)$

and has mass m_i

Then center of mass is vector sum

n

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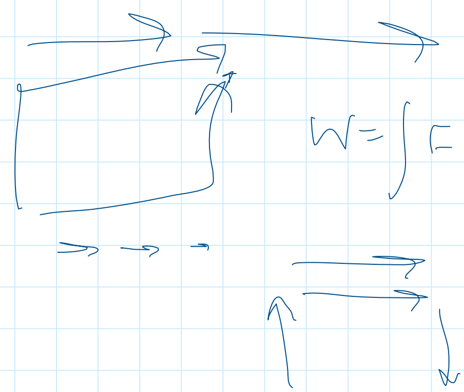
Then center of mass is vector sum

$$\frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \text{"weighted average of the locations of the objects — weighted by mass"}$$

vector sum means: x-coord of center of mass is

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$



$W = \int F \cdot dr$

y coord:

$$\frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

z-coord:

$$\frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

These formulas assume each obj has all its mass in a single point/location

realistic: mass density: $\rho(x,y,z)$ in units of mass/volume

$$\text{center of mass} = \frac{\iiint \rho(x,y,z)\,\vec{r}\;dx\,dy\,dz}{\text{total mass} = \iiint \rho(x,y,z)\,dx\,dy\,dz}$$

Q/ What does $\iiint \rho (x, y, z)$ mean?
A/ output is